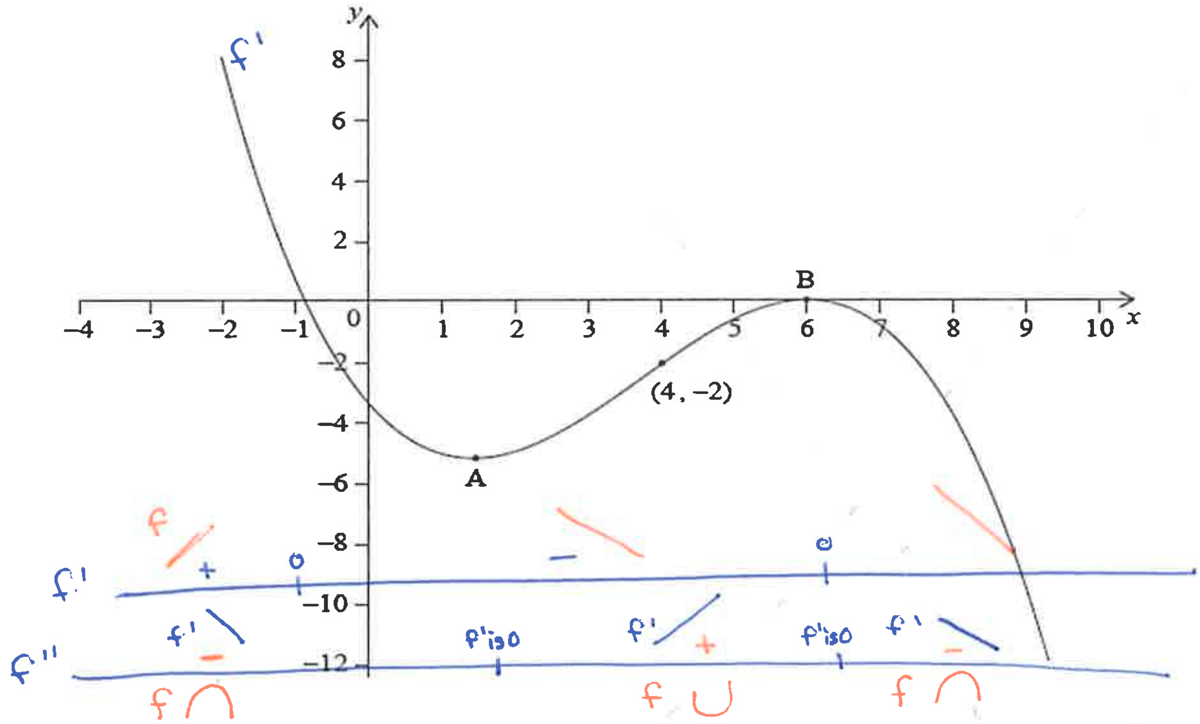
New test - January 10, 2018

1a. *[1 mark]*

The following diagram shows the graph of f', the derivative of f.



The graph of f' has a local minimum at A, a local maximum at B and passes through $(4, \ -2)$.

The point $\mathrm{P}(4, \ 3)$ lies on the graph of the function, f.

Write down the gradient of the curve of f at P.

Point on f' is $(4,-2)$ $\therefore$ gradient of f at $x=4$ is $\boxed{-2}$

1b. *[3 marks]*

$y - y_1 = m(x - x_1)$

Find the equation of the normal to the curve of f at P.

Perpendicular $\Longrightarrow$ $m = \frac{1}{2}$

$y - 3 = \frac{1}{2}(x-4)$

$y = \frac{1}{2}x - 2 + 3$

$\boxed{y = \frac{1}{2}x + 1}$

1c. *[2 marks]*

Determine the concavity of the graph of f when $4 < x < 5$ **and** justify your answer.

Since f' is increasing on $(4,5)$, f'' is positive.
If $f''>0$, then f is concave up

2. *[7 marks]*

Let $f(x) = (x^2+3)^7$. Find the term in x^5 in the expansion of the derivative, $f'(x)$.

$x(x^2)^6 + x(x^2)^5 \ \ldots \ x(x^2)^2$

x^{13} $\quad x^{11}$ $\quad x^5$

$\binom{6}{6}$ $\quad \binom{6}{5}$ $\quad \binom{6}{2}$

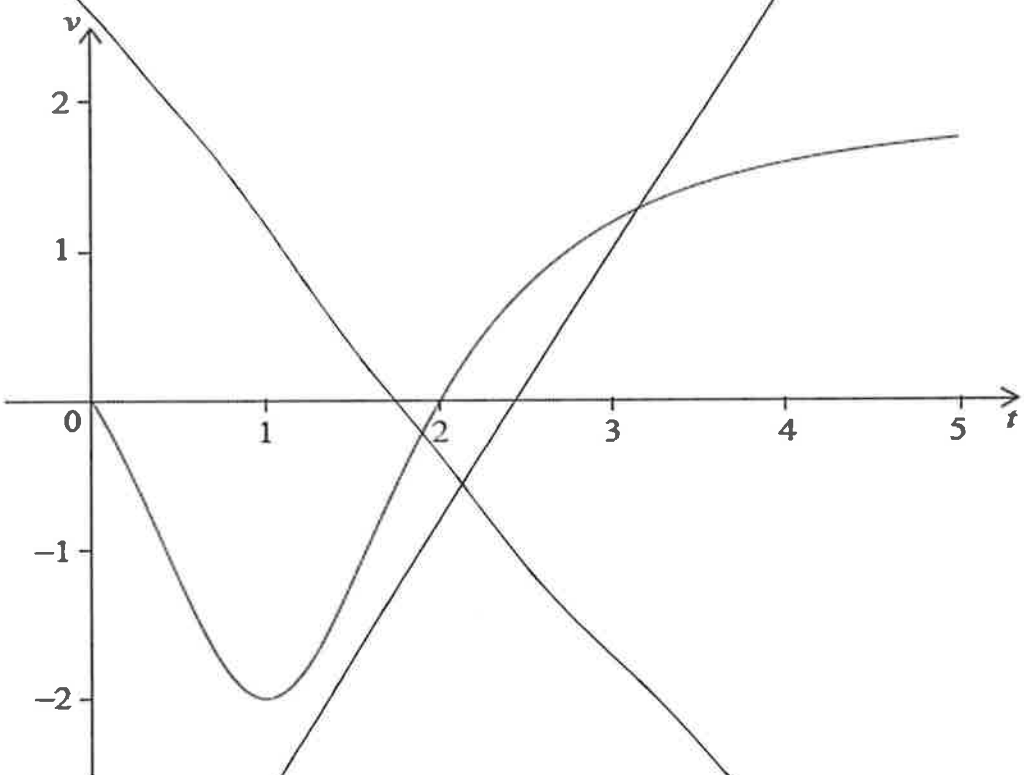
$f'(x) = 7(x^2+3)^6(2x) = 14x(x^2+3)^6$

$14x \cdot \binom{6}{2}(x^2)^2(3)^4 = 17010x^5$

3. *[6 marks]*

Note: In this question, distance is in metres and time is in seconds.

A particle moves along a horizontal line starting at a fixed point A. The velocity v of the particle, at time t, is given by $v(t) = \frac{2t^2-4t}{t^2-2t+2}$, for $0 \leqslant t \leqslant 5$. The following diagram shows the graph of v

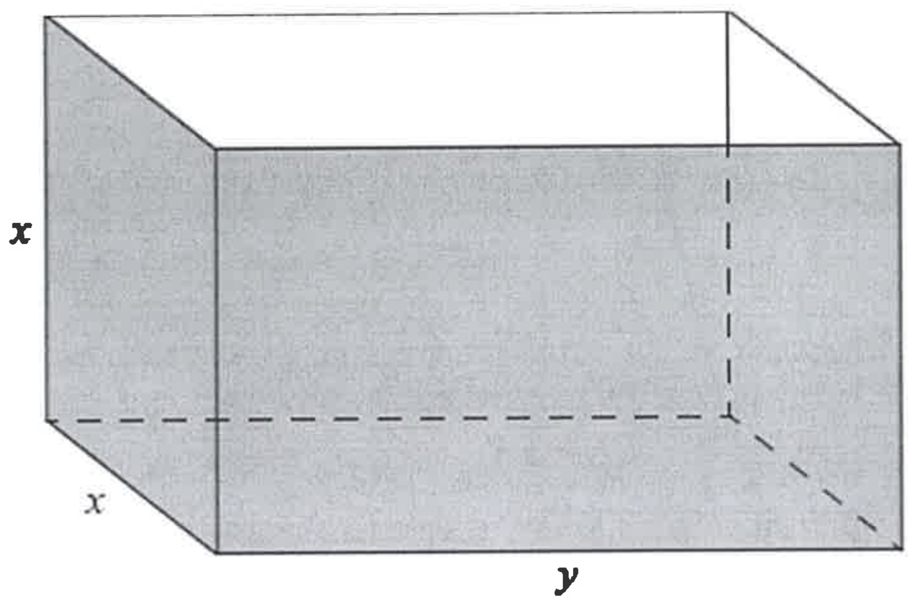


There are t-intercepts at $(0,\ 0)$ and $(2,\ 0)$.

Find the maximum distance of the particle from A during the time $0 \leqslant t \leqslant 5$ and justify your answer.

5a. *[4 marks]*

Fred makes an open metal container in the shape of a cuboid, as shown in the following diagram.



$V = x^2y = 36$

$\Rightarrow y = \frac{36}{x^2}$

$SA = 2x^2 + 2xy + 2xy$

$= 2x^2 + 4xy$

$A(x) = 2x^2 + 4x\left(\frac{36}{x^2}\right)$

$= 2x^2 + \frac{108x}{x^2}$

$A(x) = 2x^2 + \frac{108}{x}$

$\therefore A(x) = \frac{108}{x} + 2x^2$

The container has height x m, width x m and length y m. The volume is 36 m^3.

Let $A(x)$ be the outside surface area of the container.

Show that $A(x) = \frac{108}{x} + 2x^2$.

5b. *[2 marks]*

Find $A'(x)$. $= \frac{x(0) - 108(1)}{x^2} + 4x = \frac{-108}{x^2} + 4x$

5c. *[5 marks]*

Given that the outside surface area is a minimum, find the height of the container.

$\frac{-108}{x^2} + 4x = 0$

Common denom → $\frac{-108 + 4x^3}{x^2} = 0$ → $-108 + 4x^3 = 0$

$4x^3 = 108$

$x^3 = 27 \Rightarrow \boxed{x = 3}$

5d. *[5 marks]*

Fred paints the outside of the container. A tin of paint covers a surface area of $10\ m^2$ and costs \$20.

Find the total cost of the tins needed to paint the container.

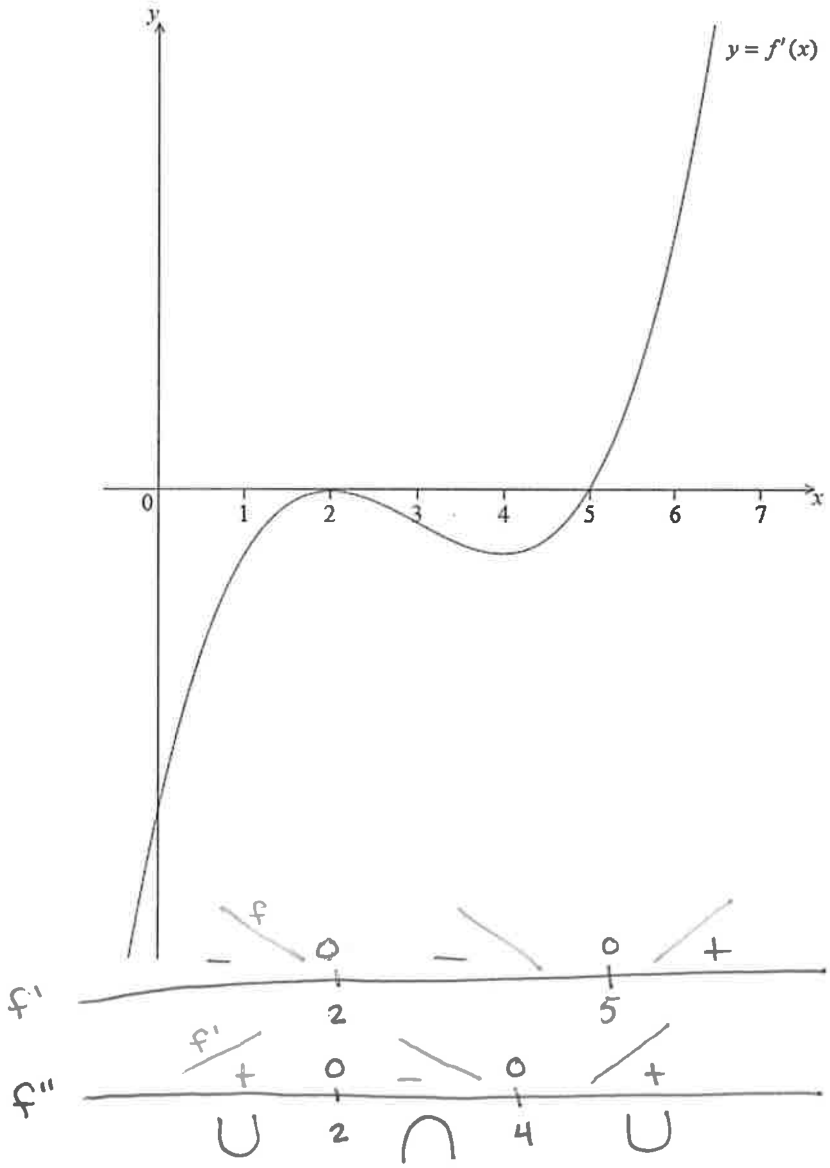
$A(3) = \frac{108}{3} + 2(3)^2 = 54 m^2$

$54 \div 10 = 5.4$ cans

use 6 cans $\times$ \$20 $= \boxed{\$120}$

6a. *[2 marks]*

Let $y = f(x)$, for $-0.5 \le x \le 6.5$. The following diagram shows the graph of f', the derivative of f.



The graph of f' has a local maximum when $x = 2$, a local minimum when $x = 4$, and it crosses the x-axis at the point $(5, 0)$.

Explain why the graph of f has a local minimum when $x = 5$. at $x=5$, $f'(x)=0$, which means that the gradient of the line tangent to f at $x=5$ is 0, which implies a local min or max on f at $x=5$.

6b. *[2 marks]*

Find the set of values of x for which the graph of f is concave down.

f is concave down on the interval $(2,4)$

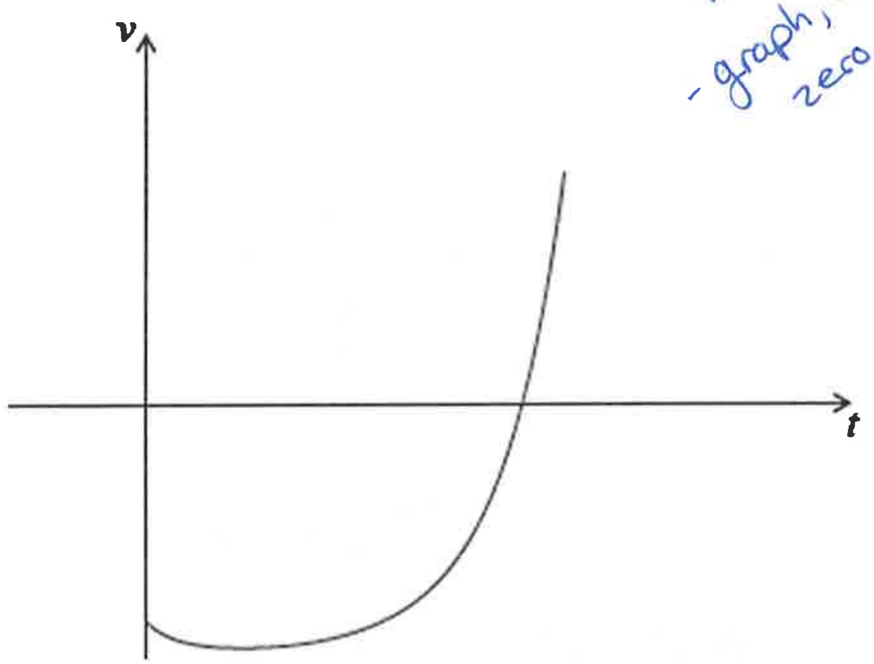
7a. *[3 marks]*

The velocity v ms^{-1} of a particle after t seconds is given by

$v(t) = (0.3t + 0.1)^t - 4$, for $0 \leq t \leq 5$

The following diagram shows the graph of v.

use a calculator to find the zero - graph, then do "calc" zero



Find the value of t when the particle is at rest. when $0 = (0.3t+0.1)^t - 4$
$t = 4.2763$ or $\boxed{4.28 \text{ seconds}}$

7b. *[3 marks]*

Find the value of t when the acceleration of the particle is 0. $a(t) = t\,(0.3t+0.1)^{t-1}$
Graph → $\boxed{t = 1.19 \text{ seconds}}$

8a. *[2 marks]*

A function f has its derivative given by $f'(x) = 3x^2 - 2kx - 9$, where k is a constant.

Find $f''(x)$. $= 6x - 2k$

$f''(x) = 6x - 2K$

8b. *[3 marks]*

The graph of f has a point of inflexion when $x = 1$.

at $x=1$, $f''(x)$ changes signs; $f''(x) = 0$ at $x=1$

Show that $k = 3$.

$0 = 6(1) - 2K$

$2K = 6$

$\boxed{K = 3}$

8c. *[2 marks]*

Find $f'(-2)$.

$f'(-2) = 3(-2)^2 - 2(3)(-2) - 9$

$= 3(4) + 12 - 9$

$= 15$

8d. *[4 marks]*

Find the equation of the tangent to the curve of f at $(-2,\ 1)$, giving your answer in the form $y = ax + b$.

at $f'(-2)$, the gradient of the tangent line is 15

so $y - 1 = 15(x+2)$

$y = 15x + 30 + 1 \implies \boxed{y = 15x + 31}$

8e. *[3 marks]*

Given that $f'(-1) = 0$, explain why the graph of f has a local maximum when $x = -1$.

$f''(x) < 0 \implies$ local max

$f''(-1) = 6(-1) - 2(3)$

$= -6 - 6 = -12$ $\quad f''(-1) < 0 \implies$ local max on $f(x)$ at $x = -1$

9a. *[2 marks]*

Let $f(x) = \frac{(\ln x)^2}{2}$, for $x > 0$.

Show that $f'(x) = \frac{\ln x}{x}$.

$f'(x) = \frac{2\left[2(\ln x)\left(\frac{1}{x}\right) - (\ln x)^2(0)\right]}{2^2} = \frac{\cancel{4}\,\frac{\ln x}{x}}{\cancel{4}} = \boxed{\frac{\ln x}{x}}$

9b. *[3 marks]*

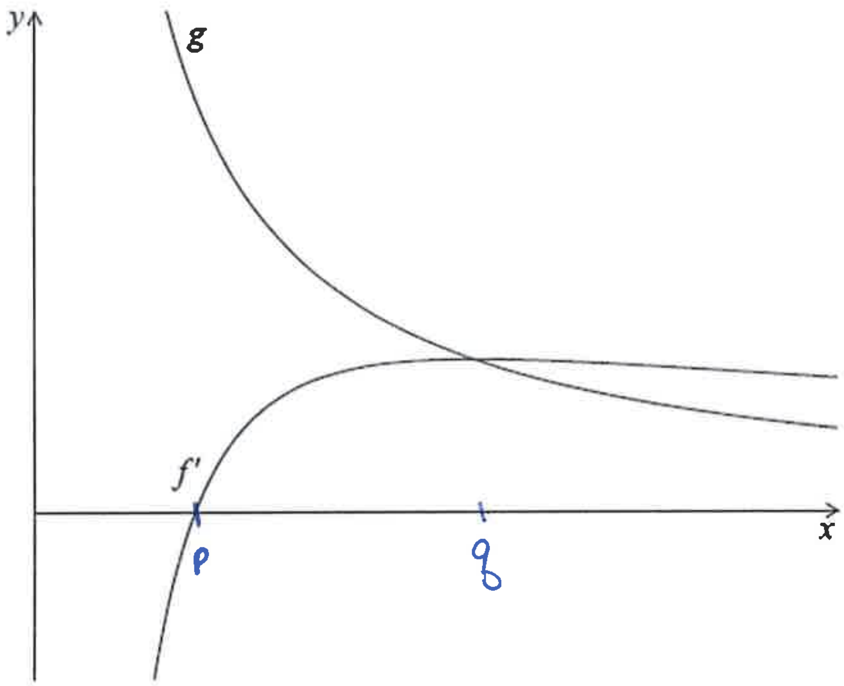
There is a minimum on the graph of f. Find the x-coordinate of this minimum.

$\cancel{x}\,\frac{\ln x}{\cancel{x}} = 0 \cdot x \implies \ln x = 0$

$e^{\ln x} = e^0 \implies \boxed{x = 1}$

9c. *[2 marks]*

Let $g(x) = \frac{1}{x}$. The following diagram shows parts of the graphs of f' and g.



The graph of f' has an x-intercept at $x = p$. $f'(x) = 0$ at $\boxed{x = 1}$

Write down the value of p.

9d. *[3 marks]*

The graph of g intersects the graph of f' when $x = q$. $x \cdot \frac{1}{x} = \frac{\ln x}{x} \cdot x$

Find the value of q.

$1 = \ln x$

$e^1 = e^{\ln x} \Rightarrow \boxed{x = e}$

10a. *[2 marks]*

Consider $f(x) = \ln(x^4 + 1)$.

Find the value of $f(0)$. $= \ln(1) = 0 \Rightarrow \boxed{f(0) = 0}$

10b. *[5 marks]*

$f'(x) = \frac{1}{x^4+1}(4x^3) = \frac{4x^3}{x^4+1}$

Find the set of values of x for which f is increasing.

$\frac{4x^3}{x^4+1} = 0$

$4x^3 = 0$

$x = 0$

$f'(-1) = \frac{-4}{2} = -2$

f': $-$ | 0 | $+$

$f'(1) = \frac{4}{2} = 2$

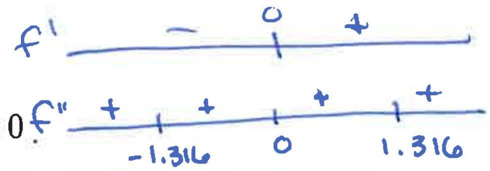
$\boxed{\text{Increasing on } x > 0}$

10c. *[5 marks]*

The second derivative is given by $f''(x) = \frac{4x^2(3-x^4)}{(x^4+1)^2}$.

The equation $f''(x) = 0$ has only three solutions, when $x = 0$, $\pm\sqrt[4]{3}$ $(\pm 1.316\ldots)$.

(i) Find $f''(1)$. $= \frac{4(2)}{4} = 2$



(ii) **Hence**, show that there is no point of inflexion on the graph of f at $x = 0$.

$f''(-2) = \frac{4(-2)^2(3-(-2))^4}{((-2)^4+1)^2}$

even power ⇒ positive

f(−1) is +

f(1) is + f(2) is +
No change in sign
⇒ no change in concavity
∴ no inflexion pt

10d. *[3 marks]*

There is a point of inflexion on the graph of f at $x = \sqrt[4]{3}$ $(x = 1.316\ldots)$.

Sketch the graph of f, for $x \geq 0$.